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MA111 - Engineering Mathematics - II
Problem Sheet - 5

Taylor Series and Maclaurin Series

1. What is the radius of convergence for the Maclaurin's series of

   (a) $\tan^{-1} x$ (b) $\log(1+x)$ (c) $\dfrac{1}{(1+x)^2}$

2. Find the Taylor's Polynomial of order 5 for the following functions.

   (a) $\tan x$ about $x = 0$
   (b) $\tan^{-1} x$ about $x = 0$
   (c) $\sin x$ about $x = \pi/3$
   (d) e^{x^2-2x} about $x = 1$

3. Find the Maclaurin's series for the following functions.

   (a) $f(x) = \dfrac{1}{1+x^2}$ (b) $f(x) = e^{\cos x}$

4. Find the Taylor's series of the following functions. Also, find the radius of convergence and interval of convergence in each case.

   (a) $\dfrac{1}{1-x}$ about $x = -1$ (b) $\dfrac{1}{2+x^2}$ about $x = 2$ (c) $\dfrac{1}{x}$ about $x = 10$

5. Using Taylor's series for $\cos x$ prove that $\dfrac{1}{2} - \dfrac{x^2}{24} < \dfrac{1-\cos x}{x^2} < \dfrac{1}{2}$, $x \neq 0$.

6. Show that the Taylor's series generated by $f(x) = \dfrac{1}{1-x}$ at $x = 0$ converges to $f(x)$ for $x \in (-1/2, 1/2)$.

7. *Approximation property of Taylor's polynomial:* Suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that

$$g(x) = b_0 + b_1(x-a) + \cdots + b_n(x-a)^n$$

   is a polynomial of degree n with constant coefficients $b_0, b_1 \ldots b_n$ and let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions (i) $E(a) = 0$ and (ii) $\lim_{x \to a} \dfrac{E(x)}{(x-a)^n} = 0$, then

$$g(x) = f(a) + f'(a)(x-a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a).$$

8. Show that a function $f(x)$ defined by a power series as $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with a positive radius of convergence R has a Taylor's series that converges to $f(x)$, for all $x \in (-R, R)$.

9. Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$. Show that

   (a) If f is even, then the Taylor's series for f at $x = 0$ contains only even powers of x. That is, $a_{2n-1} = 0$, for all $n \geq 1$.

   (b) If f is odd, then the Taylor's series for f at $x = 0$ contains only odd powers of x. That is, $a_{2n} = 0$, for all $n \geq 0$.

10. Show by using Taylor's theorem that $1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}$, for $x > 0$.

    (*Hint: By Taylor's theorem,* $\exists\ c \in (0, x)$ *such that* $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8(1+c)^{3/2}}$)

11. Show by using Taylor's theorem that $|\ln(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3})| \leq \frac{x^4}{4}$.

    (*Hint: By Taylor's theorem,* $\exists\ c \in (0, x)$ *such that* $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4(1+c)^4}$)

12. Show that for $x \in \mathbb{R}$ with $|x|^5 < \frac{5!}{10^4}$, we can replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude less than or equal to 10^{-4}.

13. Using Taylor's theorem compute $\lim_{x \to 0} \frac{1 - \sqrt{1+x^2}\cos x}{x^4}$.

    (*Hint: By Taylor's theorem,* $\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + ax^6$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + bx^5$, for some constants $a, b \in \mathbb{R}$.)

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